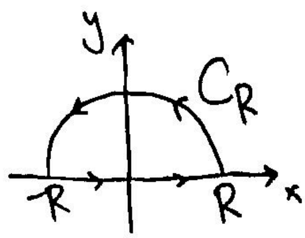
Tutorial Note Apr. 16.

1 Calculate $\displaystyle\int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+1)(x^2+2x+2)}$.



<u>Sol.</u> We consider the contour $C = [-R,R] \cup C_R$

<u>Step 1</u>. Evaluate $\displaystyle\int_C \frac{z}{(z^2+1)(z^2+2z+2)}\,dz$ for $R \gg 1$.

Poles of $f(z) = \dfrac{z}{(z^2+1)(z^2+2z+2)}$: $z_1 = i$, $z_2 = -i$, $z_3 = -1+i$, $z_4 = -1-i$

It is easy to verify that z_1, z_3, are inside the contour.

$$\operatorname{Res}(f; z_1) = \lim_{z\to z_1} (z-z_1) f(z) = \left.\frac{z}{(z+i)(z^2+2z+2)}\right|_{z=i} = \frac{1}{2(2i+1)}$$

$$\operatorname{Res}(f; z_3) = \lim_{z\to z_3} (z-z_3) f(z) = \left.\frac{z}{(z^2+1)(z+1+i)}\right|_{z=-1+i} = \frac{i-1}{2i(1-2i)}$$

We apply the Cauchy residue theorem to yield.

$$\int_C f(z)\,dz = 2\pi i\left(\operatorname{Res}(f; z_1) + \operatorname{Res}(f; z_3)\right) = -\frac{\pi}{5}$$

<u>Step 2</u>. Prove that $\displaystyle\int_{C_R} \frac{z}{(z^2+1)(z^2+2z+2)}\,dz \longrightarrow 0$ as $R \to \infty$.

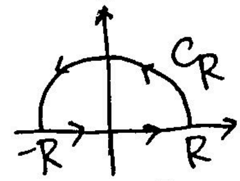
$$\left|\int_{C_R} \frac{z\,dz}{(z^2+1)(z^2+2z+2)}\right| \le \max_{z\in C_R} |f(z)|\ \text{length}(C_R) \le \frac{\pi R^2}{(R^2-1)(R^2-2R-2)} \xrightarrow{R\to\infty} 0$$

Thus, we take $R\to\infty$ to yield

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_C f(z)\,dz - \lim_{R\to\infty} \int_{C_R} f(z)\,dz = -\frac{\pi}{5}$$

$\square$

2. Prove that $\displaystyle\int_0^\infty \frac{\cos ax}{(x^2+b^2)^2}\,dx = \frac{\pi}{4b^3}(1+ab)e^{-ab} \quad (a, b > 0).$



<u>Pf.</u> We consider the contour $C(R) = [-R, R] \cup C_R$.

Then $\displaystyle\int_0^\infty \frac{\cos ax}{(x^2+b^2)^2}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos ax}{(x^2+b^2)^2}\,dx = \frac{\mathrm{Re}}{2}\left[\lim_{R\to\infty}\left(\int_{C(R)} f(z)\,dz - \int_{C_R} f(z)\,dz\right)\right]$

where $\displaystyle f(z) = \frac{e^{iaz}}{(z^2+b^2)^2}$.

<u>Step 1.</u> Calculate $\displaystyle\lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0$

$$\left|\int_{C_R} f(z)\,dz\right| \le \max_{z\in C_R}|f(z)|\ \mathrm{length}(C_R) \le \frac{\pi R}{(R^2-b^2)^2} \to 0 \text{ as } R\to\infty.$$

<u>Step 2.</u> Calculate $\displaystyle\int_{C(R)} f(z)\,dz$ for $R \gg 1$.

Poles of $f(z)$: $z_1 = bi$, $z_2 = -bi$. It is easy to verify that z_1 is inside the contour $C(R)$, and

$$\mathrm{Res}(f; z_1) = \left((z-z_1)^2 f(z)\right)'\Big|_{z=z_1} = \left(\frac{e^{iaz}}{(z+bi)^2}\right)'\Big|_{z=bi} = \frac{e^{-ab}}{4b^3 i}(ab+1)$$

We now apply the Cauchy Residue Theorem to yield

$$\int_{C(R)} f(z)\,dz = 2\pi i\,\mathrm{Res}(f; z_1) = \frac{\pi}{2b^3}e^{-ab}(ab+1).$$

Therefore,

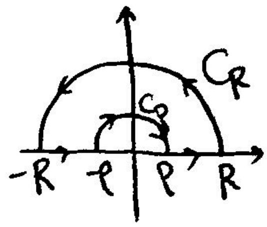
$$\int_0^\infty \frac{\cos ax}{(x^2+b^2)^2}\,dx = \frac{1}{2}\mathrm{Re}\left(\frac{\pi}{2b^3}e^{-ab}(ab+1) - 0\right) = \frac{\pi}{4b^3}(1+ab)e^{-ab}.$$

$\square$

3. Prove that $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}$

<u>Pf.</u> We consider the contour $C(\rho,R) = C_R \cup [-R,-\rho] \cup C_\rho \cup [\rho,R]$

Take $f(z) = \dfrac{z^{-\frac{1}{2}}}{z^2+1}$, for $|z|>0$ & $\arg(z) \in \left(-\frac{\pi}{2}, \frac{3}{2}\pi\right)$



Then $\displaystyle\int_{C(\rho,R)} f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{C_\rho} f(z)\,dz$

$$+ \int_{[-R,-\rho]\cup[\rho,R]} f(z)\,dz.$$

$$\int_{[-R,-\rho]} f(z)\,dz \overset{z=-x}{=\!=\!=} \int_{[R,\rho]} \frac{(-x)^{-\frac{1}{2}}}{(-x)^2+1}\,d(-x) = \int_\rho^R (-1)^{-\frac{1}{2}} \frac{x^{-\frac{1}{2}}}{x^2+1}\,dx$$

$$= \exp\left(-\frac{1}{2} i \arg(-1)\right) \int_\rho^R \frac{x^{-\frac{1}{2}}}{x^2+1}\,dx = \exp\left(-\frac{\pi}{2} i\right) \int_\rho^R \frac{x^{-\frac{1}{2}}}{x^2+1}\,dx.$$

Therefore, $\displaystyle\int_{[-R,-\rho]\cup[\rho,R]} f(z)\,dz = \left(1+\exp\left(-\frac{\pi}{2} i\right)\right) \int_\rho^R \frac{x^{-\frac{1}{2}}}{x^2+1}\,dx.$

<u>Step 1.</u> Calculate $\displaystyle\int_{C(\rho,R)} f(z)\,dz$ for $R \gg 1$ & $\rho \ll 1$.

Poles of $f(z)$ inside $C(\rho,R)$: $z_0 = i$.

$$\mathrm{Res}(f; i) = \lim_{z\to z_0} (z-z_0) f(z) = \left.\frac{z^{-\frac{1}{2}}}{z+i}\right|_{z=i} = \frac{\exp\left(-\frac{\pi}{4} i\right)}{2i}$$

Therefore, it follows from Cauchy Residue theorem that

$$\int_{C(\rho,R)} f(z)\,dz = 2\pi i\,\mathrm{Res}(f; i) = \pi \exp\left(-\frac{\pi}{4} i\right)$$

<u>Step 2.</u> Calculate $\displaystyle\lim_{R\to\infty} \int_{C_R} f(z)\,dz.$

$$\left|\int_{C_R} f(z)\,dz\right| \le \max_{z\in C_R} |f(z)|\ \mathrm{length}(C_R) \le \frac{\pi R}{R^{\frac{1}{2}}(R^2-1)} \to 0 \text{ as } R\to\infty$$

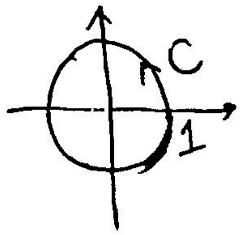
<u>Step 3.</u> Calculate $\displaystyle\lim_{\rho\to 0} \int_{C_\rho} f(z)\,dz.$

$$\left|\int_{C_\rho} f(z)\,dz\right| \le \max_{z\in C_\rho} |f(z)|\ \mathrm{length}(C_\rho) = \frac{\pi\rho}{\rho^{\frac{1}{2}}\cdot 2} \to 0 \text{ as } \rho\to 0.$$

Therefore, we take $R \to \infty$ & $\rho \to 0$ to yield

$$\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \left(1+\exp\left(-\frac{\pi}{2}i\right)\right)^{-1} \pi \exp\left(-\frac{\pi}{4}i\right) = \frac{\pi}{\sqrt{2}}. \qquad \square$$

4. Prove that $\displaystyle\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta} = \frac{2\pi}{\sqrt{1-a^2}}$, $|a|<1$.



Pf. Let $z(\theta) = e^{i\theta}$, $\theta \in [0, 2\pi]$. $\cos\theta = \dfrac{z+z^{-1}}{2}$

$$dz = ie^{i\theta}d\theta = iz\,d\theta.$$

$$\Rightarrow \int_0^{2\pi} \frac{d\theta}{1+a\cos\theta} = \int_C \frac{1}{1+a\left(\frac{z+z^{-1}}{2}\right)} \frac{1}{iz}\,dz, \text{ where } C = \{e^{i\theta}, \theta\in[0,2\pi]\}$$

Take $f(z) = \dfrac{1}{1+a\left(\frac{z+z^{-1}}{2}\right)} \dfrac{1}{iz} = \dfrac{-2i}{az^2+2z+a}$

Poles of $f(z)$: $z_1 = \dfrac{-1+\sqrt{1-a^2}}{a}$, $z_2 = \dfrac{-1-\sqrt{1-a^2}}{a}$

It is easy to verify that z_1 is inside the contour.

$$\operatorname{Res}(f; z_1) = \lim_{z\to z_1} (z-z_1)f(z) = \frac{-2i}{a} \left.\frac{1}{z+\frac{1+\sqrt{1-a^2}}{a}}\right|_{z=z_1} = \frac{1}{i\sqrt{1-a^2}}$$

Then we apply the Cauchy Residue theorem

$$\int_C f(z)\,dz = 2\pi i \operatorname{Res}(f; z_1) = \frac{2\pi}{\sqrt{1-a^2}}.$$

Therefore, $\displaystyle\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta} = \frac{2\pi}{\sqrt{1-a^2}}.$ $\square$